Wednesday, September 16, 2015

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Problem 47(a)

Problem. Determine whether the integral $\int \frac{1}{\sqrt{1-x^2}}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

Problem 47(b)

Problem. Determine whether the integral $\int \frac{x}{\sqrt{1-x^2}}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes. Let $u = 1 - x^2$ and $du = -2x\,dx$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx \\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= -\sqrt{u} + C \\ &= -\sqrt{1-x^2} + C.\end{aligned}$$

Problem 47(c)

Problem. Determine whether the integral $\int \frac{1}{x\sqrt{1-x^2}}\,dx$ can be found using the basic integration formulas studied so far.

Solution. No, we cannot yet find this integral.

Problem 48(a)

Problem. Determine whether the integral $\int e^{x^2}\,dx$ can be found using the basic integration formulas studied so far.

Solution. No, we cannot find this integral.

Problem 48(b)

Problem. Determine whether the integral $\int xe^{x^2}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes. Let $u = x^2$ and $du = 2x\,dx$. Then

$$\begin{aligned}\int xe^{x^2}\,dx &= \frac{1}{2}\int 2xe^{x^2}\,dx \\ &= \frac{1}{2}\int e^u\,du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C.\end{aligned}$$

Problem 48(c)

Problem. Determine whether the integral $\int \frac{1}{x^2}e^{1/x}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes. Let $u = \frac{1}{x}$ and $du = -\frac{1}{x^2}\,dx$. Then

$$\begin{aligned}\int \frac{1}{x^2}e^{1/x}\,dx &= -\int \left(-\frac{1}{x^2}\right) e^{1/x}\,dx \\ &= -\int e^u\,du \\ &= -e^u + C \\ &= -e^{1/x} + C.\end{aligned}$$

Problem 49(a)

Problem. Determine whether the integral $\int \sqrt{x-1}\,dx$ can be found using the basic integration formulas studied so far.

Solution. YES!!! Let $u = x - 1$ and $du = dx$. Then

$$\begin{aligned}\int \sqrt{x-1}\,dx &= \int \sqrt{u}\,du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + C\end{aligned}$$

Problem 49(b)

Problem. Determine whether the integral $\int x\sqrt{x-1}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes. Let $u = x - 1$ and $du = dx$. Then $x = u + 1$ and $dx = du$. We get

$$\begin{aligned}
\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}\,du \\
&= \int \left(u^{3/2} + u^{1/2}\right)\,du \\
&= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C.
\end{aligned}$$

Problem 49(c)

Problem. Determine whether the integral $\int \frac{x}{\sqrt{x-1}}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes. Let $u = x - 1$ and $du = dx$. Then $x = u + 1$ and $dx = du$. We get

$$\begin{aligned}
\int \frac{x}{\sqrt{x-1}}\,dx &= \int \frac{u+1}{\sqrt{u}}\,dx \\
&= \int \left(\frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}\right)\,du \\
&= \int \left(u^{1/2} + u^{-1/2}\right)\,du \\
&= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C
\end{aligned}$$

Problem 50(a)

Problem. Determine whether the integral $\int \frac{1}{1+x^4}\,dx$ can be found using the basic integration formulas studied so far.

Solution. No, not yet.

Problem 50(b)

Problem. Determine whether the integral $\int \frac{x}{1+x^4}\,dx$ can be found using the basic integration formulas studied so far.

Solution. Yes. Let $u = x^2$ and $du = 2x\,dx$. Then

$$\begin{aligned}
\int \frac{x}{1+x^4}\,dx &= \frac{1}{2}\int \frac{2x}{1+(x^2)^2}\,dx \\
&= \frac{1}{2}\int \frac{1}{1+u^2}\,du \\
&= \frac{1}{2}\arctan u + C \\
&= \frac{1}{2}\arctan x^2 + C
\end{aligned}$$

Problem 50(c)

Problem. Determine whether the integral $\int \frac{x^3}{1+x^4}\,dx$ can be found using the basic integration formulas studied so far.

Solution. YES!!! Let $u = 1 + x^4$ and $du = 4x^3\,dx$. Then

$$\begin{aligned}
\int \frac{x^3}{1+x^4}\,dx &= \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx \\
&= \frac{1}{4}\int \frac{1}{u}\,du \\
&= \frac{1}{4}\ln|u| + C \\
&= \frac{1}{4}\ln|1+x^4| + C
\end{aligned}$$

Problem 70

Problem. Consider the integral

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx.$$

(a) Find the integral by completing the square of the radicand.

(b) Find the integral by making the substitution $u = \sqrt{x}$.

(c) Determine the relationship between the two answers.

Solution. (a) Complete the square:

$$\begin{aligned} 6x - x^2 &= -(x^2 - 6x) \\ &= -(x^2 - 6x + 9) + 9 \\ &= 9 - (x-3)^2. \end{aligned}$$

Now we can integrate, using substitutions. Let $u = x - 3$ and $du = dx$. Then

$$\begin{aligned} \int \frac{1}{\sqrt{6x - x^2}}\, dx &= \int \frac{1}{\sqrt{9 - (x-3)^2}}\, dx \\ &= \int \frac{1}{\sqrt{9 - u^2}}\, du. \end{aligned}$$

Next, let $u = 3v$ and $du = 3\, dv$. Then

$$\begin{aligned} \int \frac{1}{\sqrt{9 - u^2}}\, du &= 3 \int \frac{1}{\sqrt{9 - 9v^2}}\, dv \\ &= \int \frac{1}{\sqrt{1 - v^2}}\, dv \\ &= \arcsin v + C \\ &= \arcsin \frac{u}{3} + C \\ &= \arcsin\left(\frac{x-3}{3}\right) + C. \end{aligned}$$

(b) Now let $u = \sqrt{x}$. This gets a little tricky. Rewrite this as $x = u^2$ and $dx = 2u\, du$. Also, factor $6x - x^2$ as $x(6 - x)$. (It turns out to be a BAD IDEA to complete the square, as in part (a).) Now integrate.

$$\begin{aligned} \int \frac{1}{\sqrt{6x - x^2}}\, dx &= \int \frac{1}{\sqrt{x(6-x)}}\, dx \\ &= \int \frac{2u}{\sqrt{u^2(6-u^2)}}\, du \\ &= \int \frac{2u}{u\sqrt{6-u^2}}\, du \\ &= 2\int \frac{1}{\sqrt{6-u^2}}\, du \end{aligned}$$

Let $u = \sqrt{6}v$ and $du = \sqrt{6}\,dv$. Then

$$\begin{aligned} 2\int \frac{1}{\sqrt{6-u^2}}\,du &= 2\int \frac{\sqrt{6}}{\sqrt{6-6v^2}}\,dv \\ &= 2\int \frac{1}{\sqrt{1-v^2}}\,dv \\ &= 2\arcsin v + C \\ &= 2\arcsin \frac{u}{\sqrt{6}} + C \\ &= 2\arcsin \frac{\sqrt{x}}{\sqrt{6}} + C. \end{aligned}$$

(c) The second function equals the first function minus $\frac{\pi}{2}$, so they differ by a constant.

Problem 80

Problem. An object is projected upward from ground level with an initial velocity of 500 feet per second.

(a) If air resistance is neglected, find the velocity of the object as a function of time.

(b) Use the result of part (a) to find the position function and determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of the velocity, you obtain the equation

$$\frac{dy}{dx} = -(32 + kv^2)$$

where -32 feet per second is the acceleratoin due to gravity and k is a constant. Find the velocity as a function of time by solving the equation

$$\int \frac{dv}{32 + kv^2} = -\int dt.$$

(d) Graph the velocity function for $k = 0.001$. Approximate the time t_0 at which the object reaches its maximum height.

(e) Use the integration capacities of a graphing utility to approximate the integral

$$\int_0^{t_0} v(t)\,dt.$$

Solution. (a) $a(t) = -32$, so

$$\begin{aligned} v(t) &= \int a(t)\, dt \\ &= -32t + C. \end{aligned}$$

We are given $v(0) = 500$, so it turns out that $C = 500$ and we have

$$v(t) = -32t + 500.$$

(b) Let $s(t)$ be the position function. Then

$$\begin{aligned} s(t) &= \int v(t)\, dt \\ &= \int (-32t + 500)\, dt \\ &= -16t^2 + 500t + C. \end{aligned}$$

We are given that $s(0) = 0$ (ground level), so $C = 0$ and we have

$$s(t) = -16t^2 + 500t.$$

This function reaches a maximum when $s'(t) = 0$ (i.e., $v(t) = 0$). So solve $-32t + 500 = 0$ and get

$$t_0 = \frac{500}{32} = \frac{125}{8} = 15.625.$$

(c) We must integrate $\int \frac{dv}{32+kv^2}$. Let $v = \sqrt{\frac{32}{k}}u$ and $dv = \sqrt{\frac{32}{k}}\,du$. Then we get

$$\begin{aligned}
\int \frac{dv}{32+kv^2} &= -\int dt, \\
\int \frac{\sqrt{\frac{32}{k}}\,du}{32+32u^2} &= -\int dt, \\
\frac{1}{\sqrt{32k}} \int \frac{du}{1+u^2} &= -\int dt, \\
\frac{1}{\sqrt{32k}} \arctan u &= -t + C, \\
\frac{1}{\sqrt{32k}} \arctan \sqrt{\frac{k}{32}}v &= C - t, \\
\arctan \sqrt{\frac{k}{32}}v &= \sqrt{32k}(C-t), \\
\sqrt{\frac{k}{32}}v &= \tan \sqrt{32k}(C-t), \\
v(t) &= \sqrt{\frac{32}{k}} \tan \sqrt{32k}(C-t),
\end{aligned}$$

The initial velocity is 500, so

$$C = \frac{1}{\sqrt{32k}} \arctan 500\sqrt{\frac{k}{32}}.$$

(d) If $k = 0.001$, then

$$\begin{aligned}
C &= \frac{1}{\sqrt{32(0.001)}} \arctan 500\sqrt{\frac{0.001}{32}} \\
&= 6.8603.
\end{aligned}$$

So we have

$$\begin{aligned}
v(t) &= \sqrt{\frac{32}{0.001}} \tan \sqrt{32(0.001)}(C-t) \\
&= 178.88 \tan (0.17888)(6.8603 - t).
\end{aligned}$$

The velocity will be 0 when $\tan(0.17888)(6.8603 - t) = 0$. We find

$$\begin{aligned} \tan(0.17888)(6.8603 - t) &= 0, \\ (0.17888)(6.8603 - t) &= 0, \\ 6.8603 - t &= 0, \\ t &= 6.8603. \end{aligned}$$

So, $t_0 = 6.8603$ sec.

(e) Using the TI-83, we find that

$$\int_0^{6.8603} 178.88 \tan(0.17888)(6.8603 - t)\, dt \approx 1087.96 \textbf{ ft}.$$